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April 18, 2014

Abstract

We study the structure and properties of the weak closed set of all upper bounds of a finite family of self-adjoint operators for Löwner ordering. Firstly, we prove that we can find a upper bound satisfying additional constraints. Secondly, we give two characterizations of minimal upper bounds. Finally, we furnish a complete description of pairs of positives operators such that the sum is a minimal upper bound.

Keywords: Self-adjoint operators, Löwner ordering , minimal upper bounds.
AMS Subject Class 2010. Primary 47A63, 47B65; Secondary 06A06.

1 Introduction

Let H be a separable complex Hilbert space and $B(H)$ be the Banach algebra of all continuous linear operators from H into H. Let T be in $B(H)$, we denote by $\mathcal{N}(T)$ the kernel of the operator T and by $\mathcal{R}(T)$ the range of T. We say that $T \in B(H)$ is a positive operator if T is a self-adjoint operator for which the inner product $\langle Tx|x\rangle \geqslant 0$ for all x in H. This notion of positivity induces a partial ordering on the subspace of self-adjoint operators, called Löwner order, defined as follows: for A, B in $B(H)$, we write $A \leqslant B$ if A, B are self-adjoint and $B - A$ is positive. If T is a compact operator acting on H, then $|T| = (T^*T)^{\frac{1}{2}}$ is a compact positive operator. The eigenvalues μ_1, μ_2, ... of $|T|$, arranged in decreasing order and repeated according multiplicity, form a sequence of numbers approaching 0. These numbers are called the characteristic numbers of the operator T; we write $\mu_k(T)$ for the k-th characteristic number of T. Let p be a positive real number, the Schatten class $\mathcal{S}_p(H)$ is the set of all operators such that $\sum_{k=1}^{\infty} \mu_k(T)^p < +\infty$. The function $T \longrightarrow \|T\|_p = (\sum_{k=1}^{\infty} \mu_k(T)^p)^{\frac{1}{p}}$ is a norm on $\mathcal{S}_p(H)$, and $\mathcal{S}_p(H)$ equipped with this norm is a Banach space. We denote by $\mathcal{S}_\infty(H)$ the Banach space of all compact operators. Recall that $\mathcal{S}_p(H)$ is a bilateral ideal in the algebra $B(H)$ for any $p \in]0, +\infty]$. The theory of positive operators was intensely studied by many authors (see, for instance [1], [3]). It is a crucial tool for studying a lot of problems in operator theory, especially to obtain nice inequalities and good estimates. A natural question arise in this context: What can be said about minimal upper bounds of a finite family of self-adjoint operators?

In Section 2, this investigation aims at the identification of a upper bound T of two self-adjoint operators less than the identity operator which satisfies also the constraint inequality $T \leqslant I$. More generally, we show that whenever R and S are two self-adjoint operators in a proper non-zero two sided ideal $\mathcal{I}$ of $B(H)$ such that $R, S \leq I$. Then we can find a positive operator T in $\mathcal{I}$ such that $R, S \leq T \leq I$.

The next section is devoted to study of set of all upper bounds of a finite family of self-adjoint operators. In the first part, we give a complete characterization of minimal elements in this set. We deduce a necessary and sufficient condition to ensure that the sum of the two considered self-adjoint operators is a minimal upper bound. Finally, we prove that the set of minimal upper bounds coincides with the set of extremal points of the convex set of all upper bounds of a finite family of self-adjoint operators.

In Section 4, we give a complete description of couple (R, S) of positive operators such that the sum is a minimal upper bound. The first characterization is given by a nice factorization of R and S with two orthogonal projections and a positive operator satisfying additional conditions. The second one is related to a matrix representation of R and S.

Notice that from each result related to upper bounds of a finite family of self-adjoint operators, we can easily deduce the corresponding result for the lower bounds of this family.

2 Upper bounds of two self-adjoint operators under constraints

One of the central problem in [4] is finding the so-called "natural" lower bound or upper bound of two self-adjoint operators. In the present section, by a different way, we study the existence of lower bound or upper bound under additional conditions. Let $\mathcal{I}$ be a proper non-zero two sided ideal in $B(H)$. The following result shows that we can find a maximum in $\mathcal{I}$ for two self-adjoint operators belonging in $\mathcal{I}$ satisfying an additional constraint.

Theorem 1. *Let H be an infinite dimensional Hilbert space and R, S be two self-adjoint operators in a proper non-zero two sided ideal $\mathcal{I}$ of $B(H)$ such that $R, S \leq I$. Then there exists a positive operator T in $\mathcal{I}$ such that $R, S \leq T \leq I$.*

Proof. Since $\mathcal{I}$ is a bilateral ideal of $B(H)$, using the Borelian functional calculus we can see that R and S can be decomposed under the form $R = R_+ - R_-$ and $S = S_+ - S_-$ where the four involving operators R_+, R_-, S_+, S_- are positive, less than the identity operator and belong to $\mathcal{I}$. Suppose that we can find a positive operator $T \in \mathcal{I}$ such that $R_+, S_+ \leq T \leq I$, then we clearly have $R \leq R_+ \leq T \leq I$ and $S \leq S_+ \leq T \leq I$. Thus, we have reduced the problem to the case where $0 \leq R, S \leq I$. From now on, we will assume that.

Suppose $1 \in \sigma(R)$, we set $E = \mathcal{N}(I - R)$ and we consider the matrices of R and S with respect to the direct orthogonal sum $H = E \oplus E^\perp$. We easily see that they can be written under the form

$$R = \begin{pmatrix} I_E & 0 \\ 0 & R_1 \end{pmatrix} \text{ and } S = \begin{pmatrix} A & L \\ L^* & B \end{pmatrix}.$$

If T is a positive operator such that $R, S \leq T \leq I$, it is necessarily of the form

$$T = \begin{pmatrix} I_E & 0 \\ 0 & X \end{pmatrix}.$$

We can find such T if and only if the two following conditions are satisfied:

$$R_1 \leq X \leq I_{E^\perp} \tag{1}$$

and

$$\begin{pmatrix} I_E - A & -L \\ -L^* & X - B \end{pmatrix} \geq 0. \tag{2}$$

Condition (2) is successively equivalent to:

$$\begin{cases} t^2\langle (I_E - A)x|x\rangle - 2t\,\mathrm{Re}\langle Ly|x\rangle + \langle (X-B)y|y\rangle \geq 0 \\ \forall (x,y,t) \in E \times E^{\perp} \times \mathbb{R}; \end{cases}$$

$$\begin{cases} [\langle Ly|x\rangle]^2 \leq \langle ((1+\varepsilon)I_E - A)x|x\rangle \langle (X-B)y|y\rangle \\ \forall (x,y,\varepsilon) \in E \times E^{\perp} \times \mathbb{R}_+^*; \end{cases}$$

$$\begin{cases} I_E - A \geq 0, \left\| [(1+\varepsilon)I_E - A]^{-\frac{1}{2}} Ly \right\|^2 \leq \langle (X-B)y|y\rangle \\ \forall (y,\varepsilon) \in E^{\perp} \times \mathbb{R}_+^*; \end{cases}$$

$$\begin{cases} I_E - A \geq 0, (X-B) \geq 0, \\ \forall \varepsilon > 0, L^* \left[(1+\varepsilon)I_E - A\right]^{-1} L \leq (X-B); \end{cases} \tag{3}$$

which in turn is equivalent to the following assertion:

$$I_E - A \geq 0 \text{ and } X \geq B + \lim_{\varepsilon \to 0} L^* \left[(1+\varepsilon)I_E - A\right]^{-1} L.$$

The last limit exists because a monotone function of positive operators on $\mathbb{R}_+^*$, which is uniformly norm-bounded (here by inequalities (3)), is necessarily strongly convergent at 0. We set $K = \lim L^* \left[(1+\varepsilon)I_E - A\right]^{-1} L$. Using the fact that $I - S \geq 0$, in the same manner we get

$$I_{E^{\perp}} - B \geq L^* \left[(1+\varepsilon)I_E - A\right]^{-1} L,$$

for any positive ε.

We denote by $\mathcal{I}_{E^{\perp}} = \{P_{E^{\perp}} T P_{E^{\perp}}; T \in \mathcal{I}\}$ the compression of the ideal $\mathcal{I}$, where $P_{E^{\perp}}$ stands for the orthogonal projection onto the closed subspace $E^{\perp}$. Setting $S_1 = B + K$, we see that we have to find $X \in \mathcal{I}_{E^{\perp}}$ such that

$$I_{E^{\perp}} \geq X \geq R_1, S_1,$$

where $R_1 = P_{E^{\perp}} R P_{E^{\perp}} \in B(E^{\perp})$ and $S_1 = B + K = P_{E^{\perp}} S P_{E^{\perp}} + K \in B(E^{\perp})$. Recall that a proper non-zero two sided ideal of $B(H)$ necessarily contains the space $\mathcal{F}(H)$ of all finite rank operators and is contained in the closed subspace $\mathcal{K}(H) = \mathcal{S}_{\infty}(H)$ of all compact operators (see [8], Proposition 2.1 and Corollary 2.3). On the one hand, since $S \in \mathcal{I}$, it implies that E is a finite dimensional space, hence $K \in \mathcal{F}(H) \subseteq \mathcal{I}$. Consequently, we easily see that S_1 belongs to $\mathcal{I}_{E^{\perp}}$. By straightforward computations, we can also show that $R_1 \in \mathcal{I}_{E^{\perp}}$. On the other hand, we have

$$0 \leq \|S\|\, I - S = \begin{pmatrix} \|S\|\, I_E - A & -L \\ -L^* & \|S\|\, I_{E^{\perp}} - B \end{pmatrix},$$

which leads to

$$\|S\|\, I_{E^{\perp}} \geq B + \lim_{\varepsilon \to 0} L^* \left[(\|S\| + \varepsilon)I_E - A\right]^{-1} L.$$

Observe that $\left[(\|S\| + \varepsilon)I_E - A\right]^{-1} \geq \left[(1+\varepsilon)I_E - A\right]^{-1}$, and therefore we have $\|S\|\, I_{E^{\perp}} \geq B + K = S_1$. We derive that $\|S_1\| \leq \|S\|$. Notice that $\mathcal{I}_{E^{\perp}}$ is necessarily a proper non-zero two sided ideal of $B(E^{\perp})$, so replacing R by R_1, S by S_1 and $\mathcal{I}$ by $\mathcal{I}_{E^{\perp}}$, we have reduced our problem to the case where $\|R\| < 1$, and $0 \le R, S \le I$.

Assume that $1 \in \sigma(S_1)(S_1 \to S)$, this time operators are decomposed with respect to the direct orthogonal sum $H = \mathcal{N}(I-S) \oplus \overline{\mathcal{R}(I-S)}$. Using the same process, we would find $X \in \mathcal{I}_{E^{\perp}}$ such that

$$\begin{cases} I \ge X \ge R_1, S_1 \\ \|R_1\| \le \|R\| < 1 \text{ and } \|S_1\| < 1. \end{cases}$$

Then, the problem is reduced to the case where $R, S \in \mathcal{I}$, $\|R\| < 1$ and $\|S\| < 1$. And now, let us introduce the closed subspace defined by setting

$$E_n = \bigvee_{k \le n} \left[\mathcal{N}(\mu_k(S)I - S) + \mathcal{N}(\mu_k(R)I - R)\right],$$

for every $n \ge 1$ and consider the matrices of R and S relatively to the orthogonal direct sum $H = E_n \oplus E_n^{\perp}$:

$$R = \begin{pmatrix} R_n^{'} & U_n \\ U_n^* & R_n^{''} \end{pmatrix} \text{ and } S = \begin{pmatrix} S_n^{'} & V_n \\ V_n^* & S_n^{''} \end{pmatrix}$$

We search T under the form

$$T = \begin{pmatrix} I & 0 \\ 0 & X_n \end{pmatrix}.$$

The conditions required are

$$\begin{cases} X_n \ge R_n^{''} + U_n^*(I - R_n^{'})^{-1} U_n \\ X_n \ge S_n^{''} + V_n^*(I - S_n^{'})^{-1} V_n \\ X_n \le I. \end{cases}$$

We set

$$Y_n = R_n^{''} + S_n^{''} + U_n^*(I - R_n^{'})^{-1} U_n + V_n^*(I - S_n^{'})^{-1} V_n,$$

Since $\mathcal{I}$ is a bilateral ideal, taking X_n to Y_n we see that all computations made ensure that T is in $\mathcal{I}$. Consequently, the only thing remaining to show is that Y_n could be chosen such that $\|Y_n\| < 1$.

Lemma 2. *Let T be a compact operator acting on a Hilbert space H and $(P_n)_{n \ge 0}$ a sequence of orthogonal projections which strongly converges to 0. Then the sequences $(\|P_nT\|)$ and $(\|TP_n\|)$ both converge to zero.*

Proof. Since T is a compact operator, the operator P_nT is also compact, hence we can find a unit vector x_n in H such that $\|P_nT\| = \|P_nTx_n\|$. We proceed per absurdum, suppose that $\|P_nT\|$ does not converge to 0, then there exist $\delta > 0$ and a subsequence $(x_{\varphi(n)})$, weakly convergent to some x in the closed unit ball of H, such that $\|P_{\varphi(n)}Tx_{\varphi(n)})\| \ge \delta$. Since T is a compact operator and P_n strongly converges to zero , we derive successively $\|Tx_{\varphi(n)} - Tx\| \longrightarrow 0$ and $\|P_{\varphi(n)}T\| = \|P_{\varphi(n)}Tx_{\varphi(n)}\| \le \|Tx_{\varphi(n)} - Tx\| + \|P_{\varphi(n)}Tx\| \longrightarrow 0$, a contradiction. Thus, the sequence $(\|P_nT\|)$ converges to zero. The operator T^* is also compact, hence $\|TP_n\| = \|P_nT^*\|$ also goes to zero. □

We turn now to the end of the proof of Theorem 1. By straightforward calculations it is verified that

$$\|Y_n\| \leq \left\|R_n^{''}\right\| + \left\|S_n^{''}\right\| + \frac{\|U_n\|^2}{1-\|R\|} + \frac{\|V_n\|^2}{1-\|S\|}. \tag{4}$$

Due to the positiveness of the operator S, we have $|\langle U_n y|x\rangle|^2 \leq \langle R_n^{'}x|x\rangle\langle R_n^{''}y|y\rangle$ for all $(x,y) \in E_n \times E_n^{\perp}$, hence $\|U_n\| \leq \sqrt{\|R_n^{'}\|}\sqrt{\|R_n^{''}\|}$. Similarly, we get $\|V_n\| \leq \sqrt{\|S_n^{'}\|}\sqrt{\|S_n^{''}\|}$. Then, from inequality (4) we obtain

$$\|Y_n\| \leq \left\|R_n^{''}\right\| + \left\|S_n^{''}\right\| + \frac{\left\|R_n^{'}\right\|\left\|R_n^{''}\right\|}{1-\|R\|} + \frac{\left\|S_n^{'}\right\|\left\|S_n^{''}\right\|}{1-\|S\|}$$

which can be rewritten under the form

$$\|Y_n\| \leq \|P_nRP_n\| + \|P_nSP_n\| + \frac{\|Q_nRQ_n\|\,\|P_nRP_n\|}{1-\|R\|} + \frac{\|Q_nSQ_n\|\,\|P_nSP_n\|}{1-\|S\|}, \tag{5}$$

where P_n and Q_n stand respectively for the orthogonal projections onto $E_n^{\perp}$ and E_n. Notice that P_n strongly goes to zero. Since a proper non-zero two sided ideal is necessarily contained in $\mathcal{K}(H)$, we see that R and S are compact operators, then applying Lemma 2 we deduce from inequality (5) that $\lim_{n\to\infty} \|Y_n\| = 0$. Therefore, to end the proof of Theorem 1, it suffices to choose an integer n large enough such that $\|Y_n\| < 1$ and to set $X_n = Y_n$. □

Remark 3. *Notice that the method used in the proof gives a constructive way to find such a upper bound in any proper non-zero two sided ideal of $B(H)$.*

Since every Schatten class is a proper non-zero two sided ideal of B(H), we get the following result.

Corollary 4. *Let R and S two self-adjoint operators in $\mathcal{S}_p$ $(0 < p \leq +\infty)$ such that $R, S \leq I$. Then there exists a positive operator T in $\mathcal{S}_p$ such that $R, S \leq T \leq I$.*

Corollary 5. *Let R and S two self-adjoint operators in $B(H)$ such that $R, S \leq I$. Then there exists a positive operator T in $B(H)$ such that $R, S \leq T \leq I$.*

Proof. In the finite dimensional case, we proceed as in the proof of Theorem 1, the only differences are that $\mathcal{I} = B(H)$ and that the number of steps is finite. In the infinite dimensional case, we can easily find two sequences (R_n) and (S_n) of positive operators in $\mathcal{S}_\infty = \mathcal{K}(H)$ which are respectively strongly convergent to R and S and such that $0 \leq R_n \leq I$ and $0 \leq S_n \leq I$. By previous corollary, there exists T_n in $\mathcal{K}(H)$ such that $0 \leq R_n \leq T_n \leq I$ and $0 \leq S_n \leq T_n \leq I$. Any weak limit point of the sequence (T_n) satisfies the desired conclusion. □

If S is a self-adjoint operator, we write

$$m(S) = \inf\{\lambda, \lambda \in \sigma(S)\} \text{ and } M(S) = \sup\{\lambda, \lambda \in \sigma(S)\}.$$

Corollary 6. *Let $A_1, \cdots, A_n$ be n self-adjoint operators in $B(H)$. Then there exists a minimal upper bound T in $B(H)$ such that $\sigma(T) \subseteq [\max(m(A_1), \cdots, m(A_n)), \max(M(A_1), \cdots, M(A_n))]$.*

Proof. For simplicity, we consider the case of two self-adjoint operators A, B in $B(H)$. Taking into account that the statement is translation invariant (translation by a scalar multiple of the identity), we may assume that $\max(M(A), M(B)) > 0$. Set $M = \max(M(A), M(B))$ and consider the two self-adjoint operators $A_0 = \frac{A}{M}$ and $B_0 = \frac{B}{M}$. Since $A_0, B_0 \leq I$, by applying Corollary 5 we see that there exists a operator $T_0 \in B(H)$ such that $A_0, B_0 \leq T_0 \leq I$. Thus, the operator $T_1 = MT_0$ satisfies the following constraint $A, B \leq T_1 \leq MI$. A straightforward application of Zorn's lemma with Löwner order ensures that there exists a minimal upper bound $T \in B(H)$ of A, B such that $T \leq T_0$. Therefore, we have $\max(m(A), m(B))I \leq T \leq MI$. The conclusion follows immediately. □

Remark 7. *We can remark that the previous spectral result is not valid for any minimal upper bound of two self-adjoint operators. It suffices to consider the two following matrices acting on* $\mathbb{C}^2$*:*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Combining Corollary 6, Corollary 4 from [6] and Theorem 3.2 from [7], we can obtain the next result.

Corollary 8. *Let $A_1, \cdots, A_n$ be n positive operators in $B(H)$ satisfying $\sigma(A_k) \subseteq [m, M]$ for some scalars $0 < m < M$ $(k = 1, \cdots, n)$. Let f be a increasing continuous convex function from $[m, M]$ into $\mathbb{R}_+^*$, and $\omega_k \in \mathbb{R}_+$ such that $\sum_{k=1}^n \omega_k = 1$. Then, there exist a maximal lower bound S and a minimal upper bound T of $A_1, \cdots, A_n$ such that*

$$\frac{1}{\lambda(m, M, f)} f(S) \leq \sum_{k=1}^n \omega_k f(A_k) \leq \lambda(m, M, f) f(T)$$

holds for

$$\lambda(m, M, f) = \max \left\{ f(m) + \frac{t - m}{M - m} \frac{f(M) - f(m)}{f(t)}; t \in [m, M] \right\}.$$

Proof. Since f is convex, for any $t \in [m, M]$ we have $f(t) = \sup_{i \in I} L_i(t)$ where $\{L_i; i \in I\}$ is a set of affine functions which are under the function f. Taking into account that f is increasing, we may suppose that $L_i(t) = u_i t + v_i$ with $u_i \geq 0$. We denote by E^T the spectral measure associated with T. Applying Corollary 6, we easily get the existence of a minimal upper bound T in $B(H)$ such that $\sigma(T) \subseteq [m, M]$. Let x be a unit vector and set $x_k = \sqrt{\omega_k} x$. On the one hand, applying Corollary 4 from [6], we get

$$\sum_{k=1}^n \omega_k \langle f(A_k)x|x\rangle = \sum_{k=1}^n \langle f(A_k)x_k|x_k\rangle \leq \lambda f(\sum_{k=1}^n \omega_k \langle A_k x|x\rangle)$$

where $\lambda = \lambda(m, M, f)$. On the other hand, we have

$$\begin{aligned} L_i \quad (\sum_{k=1}^n \omega_k \langle A_k x|x\rangle) &= \sum_{k=1}^n \omega_k u_i \langle A_k x|x\rangle \leq u_i \langle Tx|x\rangle + v_i \\ &= \int_m^M (u_i t + v_i) dE_{x,x}^T(t) \leq \int_m^M f(t) dE_{x,x}^T(t) = \langle f(T)x|x\rangle. \end{aligned}$$

Taking the supremum on the left side and combining these two steps, we obtain

$$\langle \left[\sum_{k=1}^{n} \omega_k f(A_k) \right] x|x\rangle \leq \lambda \langle f(T)x|x\rangle.$$

It gives the right inequality in Corollary 8.

By applying twice Jensen's inequality, we get

$$f(\sum_{k=1}^{n} \omega_k \langle A_k x|x\rangle) \leq \sum_{k=1}^{n} \omega_k f(\int_{m}^{M} t dE_{x,x}^{A_k}) \leq \sum_{k=1}^{n} \omega_k \int_{m}^{M} f(t) dE_{x,x}^{A_k} = \sum_{k=1}^{n} \omega_k \langle f(A_k)x|x\rangle. \tag{6}$$

Since $u_\alpha \geq 0$ and S is a maximal upper bound with its spectrum included in $[m, M]$, we immediately see that

$$f(\sum_{k=1}^{n} \omega_k \langle A_k x|x\rangle) \geq L_\alpha(\sum_{k=1}^{n} \omega_k \langle A_k x|x\rangle) \geq L_\alpha(\langle Sx|x\rangle).$$

Taking the supremum with respect to α, we deduce

$$f(\sum_{k=1}^{n} \omega_k \langle A_k x|x\rangle) \geq f(\langle Sx|x\rangle). \tag{7}$$

By Corollary 4 from [6], for the case of a single operator, we have

$$f(\langle Sx|x\rangle) \geq \frac{1}{\lambda} \langle f(S)x|x\rangle. \tag{8}$$

The left inequality in Corollary 8 follows directly from (6), (7) and (8). $\square$

3 Characterizations of minimal upper bounds

The following result gives a complete characterization of minimal upper bounds of a finite family of self-adjoint operators in terms of operator ranges.

Theorem 9. *Let $A_1, \ldots, A_p$ be a finite family of self-adjoint operators and T be a upper bound of $A_1, \ldots, A_p$. Then T is minimal if and only if $\mathcal{R}(\sqrt{T - A_1}) \cap \cdots \cap \mathcal{R}(\sqrt{T - A_p}) = \{0\}$.*

Proof. Suppose that a upper bound T of $A_1, \ldots, A_p$ is not minimal, then there exists an positive operator $C \neq T$ such that $0 \leq A_i \leq C \leq T$ for any $i \in \{1, \cdots, p\}$. Thus, the positive operator $R = T - C \neq 0$ satisfies the inequalities $R \leq T - A_i$ for every $i \in \{1, \cdots, p\}$. Let a be a unit vector such that $Ra \neq 0$. Observe that $(\sqrt{R}a) \otimes (\sqrt{R}a) = \sqrt{R}(a \otimes a)\sqrt{R} \leq R$. Then, we can suppose that $R = u \otimes u$ is a rank one operator. Let j be a fixed integer belonging to $\{1, \cdots, p\}$. Since $R \leq T - A_j$ we have $|\langle x|u\rangle|^2 \leq \left\| \sqrt{T - A_j} x \right\|^2$ for any $x \in H$. Let us define the operator Z_0 on $\mathcal{R}(\sqrt{T - A_j})$ by setting

$$Z_0(\sqrt{T - A_j} x) = \langle x|u\rangle \frac{u}{\|u\|}.$$

Thus, Z_0 is a contraction which could be extended on $\overline{\mathcal{R}(\sqrt{T-A_j})}$ by a contraction denoted $\widetilde{Z_0}$. Now, we define the contraction Z on $H = \mathcal{N}(\sqrt{T-A_j}) \oplus \overline{\mathcal{R}(\sqrt{T-A_j})}$ by setting $Z(a \oplus b) = \widetilde{Z_0}\, b$. Notice that $\mathcal{R}(Z) = \mathcal{R}(\widetilde{Z_0}) = \overline{\mathcal{R}(Z_0)} \subseteq \mathbb{C}u$ and hence Z is a rank one operator which can be written under the form $Z = u \otimes v$ where $\|u\|\|v\| \leq 1$ and $v \neq 0$. It follows that we have

$$\langle x|u\rangle \frac{u}{\|u\|} = Z_0\sqrt{T-A_j}x = Z\sqrt{T-A_j}x\mathrm{R} \leq R \leq T \leq I = \langle \sqrt{T-A_j}x|v\rangle u = \langle x|\sqrt{T-A_j}v\rangle u.$$

On the one hand, taking x to u, we get $\langle u|\sqrt{T-A}v\rangle u = \|u\|\, u$, which implies that $\sqrt{T-A_j}v \neq 0$. On the other hand, we have $0 \leq \|u\|^{-1} u \otimes u = Z\sqrt{T-A_j} = \sqrt{T-A_j}Z^* = \sqrt{T-A_j}v \otimes u$, saying that $\|u\|^{-1}\langle x|u\rangle u = \langle x|u\rangle\sqrt{T-A}v$ for all x in H. For $x = u$, we get $\|u\|\, u = \langle u|u\rangle\sqrt{T-A_j}v = \|u\|^2\sqrt{T-A_j}v = \|u\|\sqrt{T-A_j}v$, hence $u = \sqrt{T-A_j}v \in \mathcal{R}(\sqrt{T-A_j})$. Since j is an arbitrary element of $\{1,\cdots,p\}$, we can conclude that $\mathcal{R}(\sqrt{T-A_1}) \cap \cdots \cap \mathcal{R}(\sqrt{T-A_p}) \neq \{0\}$.

Assume that $\mathcal{R}(\sqrt{T-A_1}) \cap \cdots \cap \mathcal{R}(\sqrt{T-A_p}) \neq \{0\}$ and let u_0 be a non-null vector in the vectorial subspace $\mathcal{R}(\sqrt{T-A_1}) \cap \cdots \cap \mathcal{R}(\sqrt{T-A_p})$. We can write $u_0 = \sqrt{T-A_1}(a_1) = \cdots = \sqrt{T-A_p}(a_p)$ where $a_1,\cdots,a_p$ are non-null vectors in H. Then, we choose a positive real number t such that $t(\|a_1\| \vee \cdots \vee \|a_p\|) \leq 1$ and we set $u = tu_0 = \sqrt{T-A_1}(ta_1) = \cdots = \sqrt{T-A_p}(ta_p)$. For any $i \in \{1,\cdots,p\}$, we clearly have $u \otimes u = \sqrt{T-A_i}((ta_i) \otimes (ta_i))\sqrt{T-A_i} \leq T - A_i$. Thus, the operator $R = T - u \otimes u$ is a upper bound of $A_1,\cdots,A_p$, it is less than T and different from T. Hence T is not a minimal upper bound of $A_1,\cdots,A_p$. $\square$

Remark 10. *Let A and B be two self-adjoint operators, then $T = 1/2\,[A+B+\mid A-B\mid]$ is a concrete minimal upper bound of A and B. To see that, consider $y = \sqrt{T-A}x_1 = \sqrt{T-B}x_2 \in \mathcal{R}(\sqrt{T-A}) \cap \mathcal{R}(\sqrt{T-B})$ and decompose H into the orthogonal direct sum $H = E(\mathbb{R}_-^*) \oplus E(\mathbb{R}_+)$ where E is the spectral measure associated with $A-B$. We easily see that y necessarily belongs to $E(\mathbb{R}_-^*) \cap E(\mathbb{R}_+) = \{0\}$. Hence, we have $\mathcal{R}(\sqrt{T-A}) \cap \mathcal{R}(\sqrt{T-B}) = \{0\}$ and Theorem 9 tells us that T is a minimal upper bound. It gives an alternate proof of Corollary 5 in [2].*

In case of finite dimensional Hilbert spaces, we can give a very simple characterization.

Corollary 11. *Let $A_1,\cdots,A_p$ be a finite family of self-adjoint operators acting on a finite dimensional Hilbert space H. Then, a upper bound T of $A_1,\cdots,A_p$ is a minimal if and only if $\mathcal{N}(T-A_1) + \cdots + \mathcal{N}(T-B) = H$.*

Proof. This characterization follows directly from Theorem 9 and the equality

$$(\mathcal{R}(\sqrt{T-A_1}) \cap \cdots \cap \mathcal{R}(\sqrt{T-A_p}))^\perp = \mathcal{N}(T-A_1) + \cdots + \mathcal{N}(T-A_p)$$

which is valid in a finite dimensional Hilbert space. $\square$

Remark 12. *Notice that the natural extension of this result "$\mathcal{N}(T-A)+\mathcal{N}(T-B)$ is dense in H" does not characterizes minimal upper bounds of A and B in the infinite dimensional case. To see this, it suffices to consider the two positive operators A and B acting on $L^2[0,1]$ and defined by setting*

$$Af(x) = (1-x)f(x) \text{ and } Bf(x) = f(x) - \int_0^1 f(t)dt.$$

Then the identity operator is a minimal upper bound of A and B but the constant functions are not in the closure of $\mathcal{N}(I-A)+\mathcal{N}(I-B)$.

The sum $A+B$ is clearly the simpler example of upper bound of two positive operators A and B. A natural question is: When $A+B$ is a minimal upper bound? Theorem 9 allows us to give a complete answer.

Corollary 13. *Let A and B be two positive operators acting on H, then $A+B$ is a minimal upper bound of A and B if and only if $\mathcal{R}(\sqrt{A})\cap\mathcal{R}(\sqrt{B})=\{0\}$.*

In what follows, we denote by $\mathcal{M}_{A_1,\cdots,A_p}$ the weakly closed convex set of all upper bounds of a finite family $A_1,\ldots,A_p$ of self-adjoint operators.

Theorem 14. *Let $A_1,\ldots,A_p$ be a finite family of self-adjoint operators and $T\in\mathcal{M}_{A_1,\cdots,A_p}$. Then T is an extremal point of $\mathcal{M}_{A_1,\cdots,A_p}$ if and only if T is a minimal upper bound of $A_1,\ldots,A_p$.*

Proof. Suppose that T is not a minimal upper bound of $A_1,\ldots,A_p$. From Theorem 9, we derive that $\mathcal{R}(\sqrt{T-A_1})\cap\cdots\mathcal{R}(\sqrt{T-A_p})\neq\{0\}$. Proceeding as in the end of the proof of Theorem 9, we see that there exists a rank one operator $u\otimes u$ such that $0\neq u\otimes u\leq T-A_i$ for any $i\in\{1,\cdots,p\}$. Then, the operators $T_1=T-u\otimes u$ and $T_2=T+u\otimes u$ both belong to $\mathcal{M}_{A_1,\cdots,A_p}$. Moreover, we have $T=1/2[T_1+T_2]$ with $T_1\neq T_2$, and hence T is not an extremal point of $\mathcal{M}_{A_1,\cdots,A_p}$.

Conversely, assume that a positive operator T is not in the set $Extr(\mathcal{M}_{A_1,\cdots,A_p})$ of extreme points of $\mathcal{M}_{A,B}$. Then we can write $T=1/2(T_1+T_2)$ with T_1 and T_2 in $\mathcal{M}_{A_1,\cdots,A_p}$. Setting $R=T-T_1=T_2-T$ we see that R is a self-adjoint operator such that

$$|\langle Rx|x\rangle|\leq(\langle(T-A_1)x|x\rangle)\wedge\cdots\wedge(\langle(T-A_p)x|x\rangle). \tag{9}$$

for all $x\in H$. Then, we need the following lemma.

Lemma 15. *Let R be a self-adjoint operator acting on a Hilbert space H and S a positive operator such that*

$$|\langle Rx|x\rangle|\leq\langle Sx|x\rangle$$

for any $x\in H$. Then, we can factorize R under the form $R=\sqrt{S}J\sqrt{S}$ where J is a self-adjoint contraction.

Proof. For any positive integer n, we set $S_n=S+1/nI$. The assumption of Lemma 15 implies that $-I\leq S_n^{-\frac{1}{2}}RS_n^{-\frac{1}{2}}\leq I$. Thus, there exists a subsequence of positive integers $(\varphi(n))$ such that $S_{\varphi(n)}^{-\frac{1}{2}}RS_{\varphi(n)}^{-\frac{1}{2}}$ weakly converges to a self-adjoint contraction J. The functional calculus associated to a self-adjoint operator ensures that

$$\|\sqrt{S_{\varphi(n)}}-\sqrt{S}\|=\sup\{|\sqrt{t+\frac{1}{\varphi(n)}}-\sqrt{t}|;t\in[0,\|S\|]\}\leq\sqrt{\frac{1}{\varphi(n)}}.$$

It follows that the sequence $(S_{\varphi(n)}^{\frac{1}{2}}[S_{\varphi(n)}^{-\frac{1}{2}}RS_{\varphi(n)}^{-\frac{1}{2}}]S_{\varphi(n)}^{\frac{1}{2}})$ weakly converges to $\sqrt{S}J\sqrt{S}$, and hence $R=\sqrt{S}J\sqrt{S}$. □

We now turn to the end of the proof of Theorem 14. Using (9) and Lemma 15, we obtain that there exist self-adjoint contractions $J_1, \cdots, J_p$ such that $R = \sqrt{T - A_1}J_1\sqrt{T - A_1} = \cdots = \sqrt{T - A_p}J_2\sqrt{T - A_p}$. We immediately deduce that the non-null vectorial subspace $\mathcal{R}(R)$ is contained in $\mathcal{R}(\sqrt{T - A_1}) \cap \cdots \cap \mathcal{R}(\sqrt{T - A_p})$. Then, Theorem 9 implies that T is not a minimal upper bound of $A_1, \ldots, A_p$.

4 Description of pairs (R, S) of positive operators such that the sum $R + S$ is a minimal upper bound

In this section, we give a complete description of all pairs of positive operators for which the sum is a minimal upper bound.

Theorem 16. *Let R and S be two positive operators in $B(H)$, then the following assertions are equivalent.*

1. *The sum $R + S$ is a minimal upper bound of R and S.*
2. *There exist two orthogonal projections P_1 and P_2 with orthogonal ranges and a positive operator X such that $R = XP_1X$, $S = XP_2X$ and $\mathcal{R}(X) \subseteq \mathcal{R}(P_1+P_2) = \mathcal{R}(P_1)\oplus\mathcal{R}(P_2)$.*
3. *There exist two orthogonal subspaces E_1 and E_2 of H, a positive operator $A \in B(E_1)$, a positive operator $B \in B(E_2)$ and a bounded operator $L \in B(E_2, E_1)$ satisfying $|\langle x|Ly\rangle|^2 \leq \langle Ax|x\rangle\langle By|y\rangle$ for all $(x, y) \in E_1 \times E_2$, such that*

$$R = \begin{pmatrix} A^2 & AL & 0 \\ L^*A & L^*L & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } S = \begin{pmatrix} LL^* & LB & 0 \\ BL^* & B^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with respect to the orthogonal direct sum $H = E_1 \oplus E_2 \oplus (E_1 + E_2)^\perp$.

Proof. Firstly, we prove the implication (1) $\Rightarrow$ (3). We denote by P the orthogonal projection on $\overline{\mathcal{R}(R+S)}$ and by $Q = I - P$ the orthogonal projection on $\mathcal{N}(R + S)$. Let us introduce the operators

$$\begin{cases} \widetilde{R}_\varepsilon = (R + S + \varepsilon I)^{-\frac{1}{2}}(R + \frac{\varepsilon}{2}I)(R + S + \varepsilon I)^{-\frac{1}{2}} - \frac{1}{2}Q \\ \qquad\qquad\text{and} \\ \widetilde{S}_\varepsilon = (R + S + \varepsilon I)^{-\frac{1}{2}}(S + \frac{\varepsilon}{2}I)(R + S + \varepsilon I)^{-\frac{1}{2}} - \frac{1}{2}Q, \end{cases} \tag{10}$$

where ε is any positive real number. Observe that $\widetilde{R}_\varepsilon + \widetilde{S_\varepsilon} = I - Q = P$, therefore we see that $\widetilde{R}_\varepsilon$ is a positive contraction. Let us show that $\widetilde{R}_\varepsilon$ (resp. $\widetilde{S_\varepsilon}$) are weakly convergent.

Let $x = (R + S)^{\frac{1}{2}}a$ and $y = (R + S)^{\frac{1}{2}}b$ be in $\mathcal{R}(R + S)^{\frac{1}{2}}$. On the one hand, we have

$$\langle\widetilde{R}_\varepsilon x|y\rangle = \langle(R + \frac{\varepsilon}{2}I)(R + S + \varepsilon I)^{-\frac{1}{2}}(R + S)^{\frac{1}{2}}a|(R + S + \varepsilon I)^{-\frac{1}{2}}(R + S)^{\frac{1}{2}}b\rangle.$$

On the other hand, setting $A = R + S$ and denoting by E^A the spectral measure associated with A, we see that

$$\left\|(A + \varepsilon I)^{-\frac{1}{2}}A^{\frac{1}{2}}a - Pa\right\|^2 = \int\limits_{]0,\|A\|]} \frac{\varepsilon^2}{(\sqrt{t} + \sqrt{t+\varepsilon})^2(t + \varepsilon)}dE^A_{a,a}(t) \to 0$$

in virtue of the dominated convergence theorem. Therefore, $\langle \widetilde{R}_\varepsilon x|y\rangle$ converges for any $x, y \in \mathcal{R}((R+S)^{\frac{1}{2}})$. Now, if $x \in \mathcal{N}(R+S)$, we have $\widetilde{R}_\varepsilon x = 0$. Let $\delta > 0$ and $x, y \in \overline{\mathcal{R}(R+S)^{\frac{1}{2}}} = \overline{\mathcal{R}(R+S)}$ such that $\|x - x'\| \vee \|y - y'\| \leq \delta$ with $x', y' \in \mathcal{R}((R+S)^{\frac{1}{2}})$, then a straightforward computation leads to

$$|\langle \widetilde{R}_{\varepsilon_1} x|y\rangle - \langle \widetilde{R}_{\varepsilon_2} x|y\rangle| \leq |\langle (\widetilde{R}_{\varepsilon_1} - \widetilde{R}_{\varepsilon_2})x^{'}|y^{'}\rangle| + 2\delta^2 + 2\delta(\|x\| + \delta) + 2\delta(\|y\| + \delta)$$

Since $|\langle (\widetilde{R}_{\varepsilon_1} - \widetilde{R}_{\varepsilon_2})x^{'}|y^{'}\rangle \to 0$ when $\varepsilon \to 0$, we see that $(\langle \widetilde{R}_\varepsilon x|y\rangle)$ is a Cauchy sequence, hence is convergent. Finally, the sequence $(\langle \widetilde{R}_\varepsilon x|y\rangle)$ converge for any $x, y \in H$. Thus, the uniformly bounded operator function $(\widetilde{R}_\varepsilon)$ (resp. $(\widetilde{S}_\varepsilon)$) weakly converges to some positive operator $\widetilde{R}$ (resp. $\widetilde{S}$). Taking the limit when ε goes to 0, we easily obtain from (10) the following equalities: $R = \sqrt{R+S}\widetilde{R}\sqrt{R+S}$ and $S = \sqrt{R+S}\widetilde{S}\sqrt{R+S}$. We also get $\widetilde{R} + \widetilde{S} = P$.

Let us show that P is a minimal upper bound for $\widetilde{R}$ and $\widetilde{S}$. Suppose that there exists a positive operator $J \neq P$ such that $\widetilde{R}, \widetilde{S} \leq J \leq P$. It leads to $R = \sqrt{R+S}\widetilde{R}\sqrt{R+S} \leq \sqrt{R+S}J\sqrt{R+S} \leq R+S$, and similarity $S \leq R+S$. Since $R+S$ is minimal, we have necessarily $R+S = \sqrt{R+S}J\sqrt{R+S}$, which in turn implies $\sqrt{R+S}(P-J)\sqrt{R+S} = 0$. It follows that $(P-J)\sqrt{R+S} = 0$ and finally $P = J$ because the positive operator $P - J$ is null on $\mathcal{N}(R+S)$.

On the one hand, Corollary 13 gives that $\mathcal{R}(\sqrt{\widetilde{R}}) \cap \mathcal{R}(\sqrt{\widetilde{S}}) = \{0\}$. On the other hand, the inequalities $0 \leq \widetilde{R} \leq P$ successively imply $Q\widetilde{R}Q = 0$, $\sqrt{\widetilde{R}}Q = 0$, and hence $\widetilde{R}P = \widetilde{R} = \widetilde{R}^* = P\widetilde{R}$. Let $x \in H$, we then have $\widetilde{R}x - \widetilde{R}^2 x = P\widetilde{R}x - \widetilde{R}^2 x = \widetilde{S}\widetilde{R}x \in \mathcal{R}(\sqrt{\widetilde{R}}) \cap \mathcal{R}(\sqrt{\widetilde{S}}) = \{0\}$, thus $\widetilde{R} = \widetilde{R}^2$. In a similar way we prove that $\widetilde{S} = \widetilde{S}^2$, hence $\widetilde{R}$ and $\widetilde{S}$ are two orthogonal projections. Set $E_1 = \mathcal{R}(\widetilde{R})$ and $E_2 = \mathcal{R}(\widetilde{S})$, we have

$$\widetilde{R} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \widetilde{S} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the orthogonal sum $H = E_1 \oplus E_2 \oplus (E_1 + E_2)^\perp$. The matrix of the positive operator $\sqrt{R+S}$ is necessarily of the form

$$\sqrt{R+S} = \begin{pmatrix} A & L & 0 \\ L^* & B & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $A \in B(E_1)$ and $B \in B(E_2)$ are positive operators and $L \in B(E_2, E_1)$ is a a bounded operator satisfying $|\langle x|Ly\rangle|^2 \leq \langle Ax|x\rangle\langle By|y\rangle$ for all $(x, y) \in E_1 \times E_2$ because of the positiveness of $\sqrt{R+S}$. The equalities $R = \sqrt{R+S}\widetilde{R}\sqrt{R+S}$ and $S = \sqrt{R+S}\widetilde{S}\sqrt{R+S}$ give the desired matrix representations of R and S.

Concerning the implication (3) $\Rightarrow$ (2), we have just to set $X = \sqrt{R+S}$, $P_1 = \widetilde{R}$ and $P_2 = I - \widetilde{R}$ (with the notations used in the proof of (1) $\Rightarrow$ (3)). Notice that the property: $|\langle x|Ly\rangle|^2 \leq \langle Ax|x\rangle\langle By|y\rangle$ for all $(x, y) \in E_1 \times E_2$, ensures that the self adjoint operator X is positive. By construction, the subspaces $Im(P_1)$ and $Im(P_2)$ are contained in Im(X).

Let us now prove the implication (2) $\Rightarrow$ (1). Since $R = XP_1X$ and $S = XP_2X$, we have $R + S = XP_1X + XP_2X = X(P_1 + P_2)X = X^2$. The last equality is due to the inclusion $\mathcal{R}(X) \subseteq \mathcal{R}(P_1 + P_2)$ and the fact that $P_1 + P_2$ is necessarily an orthogonal projection. We thus have $X = \sqrt{R+S}$. We suppose that L is a upper bound of R and S such that $L \leq R+S = X^2$.

Notice that $\|\sqrt{L}x\| \leq \|Xx\|$. According to the well known criterion of Douglas about range inclusion and factorization of operators (see [5] for more informations), we see that there exists a contraction $Y \in B(H)$ such that $\sqrt{L} = YX$ and $KerY = KerX$. Let $x = Xx_1 + x_0$ where $x_1 \in H$ and $x_0 \in KerX$, we have

$$\begin{aligned}\langle Y^*Yx|x\rangle &= \langle Y^*Y(Xx_1 + x_0)|Xx_1 + x_0\rangle = \langle YXx_1|YXx_1\rangle \\ &= \langle Lx_1|x_1\rangle \geq \langle P_1Xx_1|Xx_1\rangle = \langle P_1x|x\rangle.\end{aligned}$$

We derive that $Y^*Y \geq P_1$ and in the same manner we can prove that $Y^*Y \geq P_2$. Therefore, for any $x \in \mathcal{R}(P_1)$ we have $\|x\|^2 = \|P_1x\|^2 \leq \|Yx\|^2 \leq \|x\|^2$, hence $Y^*Yx = x$. Similarly, $Y^*Yx = x$ for any $x \in \mathcal{R}(P_2)$. Thus, we have $Y^*Yx = x$ for every $x \in \mathcal{R}(P_1) \oplus \mathcal{R}(P_2) \supseteq \mathcal{R}(X)$. Finally, we can conclude that $L = (\sqrt{L})^*(\sqrt{L}) = (YX)^*(YX) = XY^*YX = X^2 = R + S$. Hence $R + S$ is minimal. $\square$

The second author wishes to note that the original idea of this paper is due to the first author.

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